Anomalous scaling of passively advected magnetic field in the presence of strong anisotropy

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Inertial-range scaling behavior of high-order (up to order $N=51$) two-point correlation functions of a passively advected vector field has been analyzed in the framework of the rapid-change model with strong small-scale anisotropy with the aid of the renormalization group and the operator-product expansion. Exponents of the powerlike asymptotic behavior of the correlation functions have been calculated in the one-loop approximation. These exponents are shown to depend on anisotropy parameters in such a way that a specific hierarchy related to the degree of anisotropy is observed. Deviations from power-law behavior like oscillations or logarithmic behavior in the corrections to correlation functions have not been found.

I. INTRODUCTION

Justification of the basic principles of the Kolmogorov-Obukhov (KO) phenomenological theory [1–3] and the investigation of possible deviations from its conclusions within the framework of a microscopic model is one of the main tasks in the theory of the fully developed turbulence and related models, e.g., stochastic magnetohydrodynamics (MHD).

According to the KO theory, the following single-time structure functions in the inertial range $(r_d \ll r \ll r_l)$

$$S_N(r) \equiv \langle [v_r(\mathbf{x},t) - v_r(\mathbf{x}',t)]^N \rangle, \quad r = |\mathbf{x} - \mathbf{x}'| \quad (1)$$

are independent of both the external (integral) scale r_l and internal (viscous) scale r_d, the latter being tantamount to independence of viscosity. These requirements are the famous first and second hypotheses of Kolmogorov, respectively. In Eq. (1) v_r denotes the component of the velocity field directed along the separation vector $\mathbf{r}=\mathbf{x}-\mathbf{x}'$. Dimensional arguments then determine the scale-invariant form of the structure functions (1) as

$$S_N(r) = \text{const} \times (\bar{\epsilon} r)^{N/3},$$

where $\bar{\epsilon}$ is the mean dissipation rate [1–3].

On the other hand, both theoretical and experimental results reveal some deviations from the KO theory [1,4], viz. contradiction with the first Kolmogorov hypothesis. For the structure functions (1) it means that they have to be modified in the following way:

$$S_N(r) = (\bar{\epsilon} r)^{N/3} \zeta_N(r/r_l), \quad (2)$$

where $\zeta_N(r/r_l)$ are scaling functions with powerlike behavior in the asymptotic region $r/r_l \ll 1$

$$\zeta_N(r/r_l) \simeq \text{const} \times \left(\frac{r}{r_l}\right)^{q_N}. \quad (3)$$

The singular dependence of the structure functions on r_l in the limit $r_l \rightarrow \infty$ together with nonlinearity of the exponents q_N as functions of N is called "anomalous scaling." Theoretical explanation of such behavior is based on strongly developed fluctuations of the dissipation rate, i.e., intermittency [1,3].

A suitable and also powerful method of studying self-similar scaling behavior is that of the renormalization group (RG) [5–8]. In the theory of critical phenomena it successfully explains the origin of the critical scaling. This technique is also applicable to the theory of turbulence, see Refs. [6,8,9], and references therein.

The traditional approach to the description of fully developed turbulence is based on the stochastic Navier-Stokes equation [10]. Within the RG method the second Kolmogorov hypothesis (independence of the viscous length r_d) was proved for structure functions (1) for a variety of realistic random forces (see Ref. [8]). This fact leads to the existence of the IR scaling $(r \gg r_d)$ with definite "critical dimensions"

$$\Delta[S_N] = -\frac{N}{3} \quad (4)$$

with the customary convention $\Delta[r]=-1$ for scaling dimensions. In the framework of the RG approach, critical dimensions such as Eq. (4) arise as coefficients in the differential RG equations. On the other hand, it is not possible to infer the form of the scaling functions $\zeta_N(r/r_l)$ in Eq. (2) from the RG equations.

The standard way to investigate the dependence of the scaling functions on the argument r/r_l in the limit case r/r_l

$\rightarrow 0$ is the utilization of the operator-product expansion (OPE), see Refs. [5,6,8]. The OPE leads to the following representation of scaling functions (3):

$$\zeta_N(r/r_l) = \sum_F C_F \left(\frac{r}{r_l}\right)^{\Delta_F}, \quad r/r_l \rightarrow 0, \qquad (5)$$

where summation over all possible composite operators F (i.e., products of fields and their derivatives) is implied (for details, see Sec. IV), Δ_F are their critical dimensions, and coefficients C_F are regular functions of r_l^{-1}.

Contrary to the theory of critical phenomena, where there are physical reasons to believe that all relevant composite operators have positive critical dimensions ($\Delta_F > 0$) for physical values of parameters [this is why the leading term in the expansion (5) is given by the trivial operator $F=1$ ($\Delta_1 = 0$) in that case], in the theory of fully developed turbulence based on the stochastic Navier-Stokes equation critical dimensions of many composite operators are definitely negative for physical values of parameters. The existence of these "dangerous operators" leads to singular behavior of structure functions in the limit $r/r_l \rightarrow 0$ [8]. In the stochastic Navier-Stokes model dangerous operators enter into the OPE in the form of infinite families with the spectrum of critical dimensions unbounded from below, and a nontrivial problem of the summation of their contributions arises. This is an unsolved problem of the theory.

In a situation where there are difficulties to study the anomalous scaling in the stochastic Navier-Stokes model (this applies to the stochastic MHD as well) it does not seem to be unreasonable to consider simpler models, which have features similar to real turbulent flow and, on the other hand, are easier for investigation. An important role in this study was played by the model of passive advection of a scalar quantity (temperature or concentration of the tracer) by a uncorrelated-in-time Gaussian velocity field [11]. Models of passively advected vector fields [12] are straightforward generalizations of the model of the passive advection of a scalar field.

In the present work the spatial structure of correlations of fluctuations of the magnetic (vector) field **b** in a given turbulent fluid in the framework of the kinematic MHD Kazantsev-Kraichnan model (KMHD) is studied. These fluctuations are generated stochastically by a Gaussian random emf and a white in time and anisotropic self-similar in space Gaussian drift. The main goal is the calculation of the anomalous exponents as functions of the anisotropy parameters of the drift. Here, numerical calculation of the critical dimensions Δ_F in the one-loop approximation has been extended to dimensions related to correlation functions of order $N = 51$ to explore possible departures from powerlike asymptotic behavior.

II. KINEMATIC MHD KAZANTSEV-KRAICHNAN MODEL

Consider passive advection of a solenoidal magnetic field $\mathbf{b} \equiv \mathbf{b}(\mathbf{x},t)$ in the framework of the KMHD model described by the stochastic equation

$$\partial_t \mathbf{b} = \nu_0 \Delta \mathbf{b} - (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{b} + (\mathbf{b} \cdot \boldsymbol{\nabla}) \mathbf{v} + \mathbf{f}, \qquad (6)$$

where $\partial_t \equiv \partial/\partial t$, $\Delta \equiv \boldsymbol{\nabla}^2$ is the Laplace operator, ν_0 is the coefficient of the magnetic diffusivity, and $\mathbf{v} \equiv \mathbf{v}(\mathbf{x},t)$ is a random solenoidal (owing to the incompressibility) velocity field. Thus, both **v** and **b** are divergence-free vector fields $\boldsymbol{\nabla} \cdot \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{b} = 0$. A transverse Gaussian emf flux density **f** $\equiv \mathbf{f}(\mathbf{x},t)$ with zero mean and the correlation function

$$D_{ij}^f \equiv \langle f_i(\mathbf{x},t) f_j(\mathbf{x}',t') \rangle = \delta(t-t') C_{ij}(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}' \qquad (7)$$

is the source of the fluctuations of the magnetic field **b**. The parameter L represents an integral scale related to the stirring, and C_{ij} is a function finite in the limit $L \rightarrow \infty$. In the present treatment its precise form is irrelevant, and with no loss of generality, we take $C_{ij}(0) = 1$ in what follows. The random velocity field **v** obeys Gaussian statistics with zero mean and the correlation function

$$D_{ij}^v(\mathbf{x},t) \equiv \langle v_i(\mathbf{x},t) v_j(0,0) \rangle = \frac{D_0 \delta(t)}{(2\pi)^d} \int d^d\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{x}} T_{ij}(\mathbf{k})}{(k^2 + r_l^{-2})^{d/2+\epsilon/2}}, \qquad (8)$$

where r_l is another integral scale. In general, the scale r_l may be different from the integral scale L, below we, however, take $r_l \simeq L$. $D_0 > 0$ is an amplitude factor related to the coupling constant g_0 of the model by the relation $D_0/\nu_0 \equiv g_0 \equiv \Lambda^\epsilon$, where Λ is the characteristic UV momentum scale, and $0 < \epsilon < 2$ is a free parameter. Its "physical" value $\epsilon = 4/3$ corresponds to the Kolmogorov scaling of the velocity correlation function in developed turbulence. d is the dimensionality of the coordinate space. In the isotropic case, the second-rank tensor $T_{ij}(\mathbf{k})$ in Eq. (8) has the simple form of the ordinary transverse projector $T_{ij}(\mathbf{k}) = P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j / k^2$.

Although the structure functions $S_N(r)$ of the magnetic field defined in analogy with Eq. (1) as

$$S_N(r) \equiv \langle [b_r(\mathbf{x},t) - b_r(\mathbf{x}',t)]^N \rangle, \quad r \equiv |\mathbf{x} - \mathbf{x}'|, \qquad (9)$$

are important tools in the analysis of MHD turbulence in the inertial range [defined by the inequalities $r_d \ll r \ll r_l$, where $r_d \simeq \Lambda^{-1}$ is an internal (viscous) scale], we shall here concentrate on simpler quantities, i.e., the equal-time two-point correlation functions [equal-time correlation functions of the "composite operators" $b_r^{N-m}(\mathbf{x},t)$ and $b_r^m(\mathbf{x},t)$]

$$B_{N-m,m}(r) = \langle b_r^{N-m}(\mathbf{x},t) b_r^m(\mathbf{x}',t) \rangle, \quad r \equiv |\mathbf{x} - \mathbf{x}'| \qquad (10)$$

for two reasons: first, the field-theoretic approach yields the scaling behavior of these quantities in the first place, while the scaling behavior of the structure functions (9) emerges from their representation as linear combinations of the two-point correlation functions (10). Second, contrary to the problems of turbulent velocity of incompressible fluid and passive scalar advected by such fluid, the basic stochastic equation (6) is not invariant under the shift $\mathbf{b} \rightarrow \mathbf{b} + \mathbf{c}$, where **c** is a constant vector. Thus, there is no compelling need to aim at the analysis of more complex quantities, the structure functions, instead of their building blocks, the correlation functions (10).

Dimensional analysis yields

$$B_{N-m,m}(r) = \nu_0^{-N/2} r^N R_{N,m}(r/r_d, r/r_l), \qquad (11)$$

where $R_{N,m}$ are functions of dimensionless parameters. When the random source field $\mathbf{f}$ and the velocity field $\mathbf{v}$ are uncorrelated, the correlation functions of odd order $B_{2n+1-m,m}$ vanish, however. The standard perturbation expansion (series in g_0) is ill suited for calculation of correlation functions (11) in the limit $r/r_d \to \infty$ and $r/r_l \to 0$, due to the singular behavior of the coefficients of the expansion. Therefore, to find the correct IR behavior it is necessary to sum the whole series. Such a summation can be carried out within the field-theoretic RG and OPE. A compact description of this procedure is presented in Refs. [13–15] (see also Ref. [16]). Below we remind basic ideas and results referring to the isotropic case for simplicity of notation.

The RG analysis can be divided into two main parts. First, the UV renormalization of correlation functions (10) is carried out. As a consequence of this the asymptotic behavior of these functions for $r/r_d \gg 1$ and arbitrary but fixed r/r_l is given by IR stable fixed point(s) of the corresponding RG equations and for correlation functions (10) the following asymptotic form is obtained:

$$B_{N-m,m}(r) \sim \nu_0^{-N/2} r_d^N \left(\frac{r}{r_d}\right)^{-\Delta[B_{N-m,m}]} \xi_{N,m}(r/r_l) \quad \frac{r}{r_d} \gg 1,$$
$$(12)$$

where the critical dimensions $\Delta[B_{N-m,m}]$ are expressed in terms of the "anomalous dimensions" γ_ν^* and γ_N^* of the viscosity ν and the composite operators b_r^N, respectively, as

$$\Delta[B_{N-m,m}] = -N\left(1 - \frac{\gamma_\nu^*}{2}\right) + \gamma_{N-m}^* + \gamma_m^*. \qquad (13)$$

The scaling functions $\xi_{N,m}(r/r_l)$ in relations (12) remain unknown. The critical dimensions $\Delta[B_{N-m,m}]$ are calculated as (asymptotic) series in ϵ with the use of renormalized perturbation theory, and this is why the exponent ϵ here plays the role analogous to the parameter $4-d$ in the RG theory of critical phenomena. Another parallel is related to the parameter r_l which is an analog of the correlation length r_c [5,6].

Second, the small r/r_l behavior of the functions $\xi_{N,m}(r/r_l)$ has to be estimated. This may be done using the OPE, which leads to the following asymptotic form in the limit $r/r_l \to 0$:

$$\xi_{Nm}(r/r_l) = \sum_F C_F(r/r_l)\left(\frac{r}{r_l}\right)^{\Delta_F},$$

where $C_F(r/r_l)$ are coefficients regular in r/r_l. The summation is implied over all possible renormalized scale-invariant composite operators F and Δ_F are their critical dimensions.

In the limit $L/r \to \infty$ correlation function (7) of the random source field is uniform in space, which—as usual in stochastic models describing passive transport [13–16]—brings about composite operators with negative critical dimensions (dangerous composite operators) at the outset in the asymptotic analysis. This takes place because the limiting behavior of the correlation function determines the "canonical" scaling dimension of the magnetic field which in this

case becomes equal to -1. Origin of the dangerous operators is thus different from that of the stochastic Navier-Stokes problem, where canonical field dimensions are positive and composite operators become dangerous (i.e., acquire negative scaling dimension) only for large enough values of the RG expansion parameter [7,8].

The velocity fluctuation contribution to the scaling dimension in the passive transport problems is independent of the statistical properties of the source field. It is important to bear this in mind, because below it will be shown that calculation of the fluctuation corrections in the present problem is very similar to that in the case of passively advected scalar if the magnetic field $\mathbf{b}$ is traded for the vector field $\nabla\theta$—the gradient of the scalar. This occurs because—as a consequence of the invariance of the transport equation for the scalar θ with respect to the shift $\theta \to \theta + c$ with any constant c—in the scalar problem fluctuation corrections to scaling behavior are determined by composite operators constructed not from the scalar field itself but its derivatives $\nabla\theta$. Physically, however, these problems are different, because the usual random source for the scalar with a correlation function $C(r/L) \to 1$ in the limit $r/L \ll 1$ corresponds to random source for the vector field $\nabla\theta$ with correlations concentrated at small separations (large wave numbers) instead of the asymptotically flat correlation function in the coordinate space (corresponding to strong correlation at small wave numbers in the wave-vector space).

Contributions of these dangerous operators with negative scaling dimensions to the OPE imply singular behavior of the scaling functions in the limit $r/r_l \to 0$. The leading term is given by the operator with the most negative critical dimension Δ_F. In our model the leading contributions to correlation functions of even order $B_{N-m,m}$ ($N=2n$) are given by scalar operators $F_N = (\mathbf{b} \cdot \mathbf{b})^{N/2}$ with their critical dimensions $\Delta_N = -N(1 - \gamma_\nu^*/2) + \gamma_N^*$, which eventually determine the nontrivial asymptotic behavior of the correlation functions $B_{N-m,m}$ of the form (the correlation function $B_{N,0} = B_{0,N}$ is a constant)

$$B_{N-m,m}(r) \sim \nu_0^{-N/2} r_l^N \left(\frac{r_d}{r_l}\right)^{N\gamma_\nu^*/2} \left(\frac{r}{r_d}\right)^{-\gamma_{N-m}^* - \gamma_m^*} \left(\frac{r}{r_l}\right)^{\gamma_N^*}$$
$$\sim r^{\gamma_N^* - \gamma_{N-m}^* - \gamma_m^*}. \qquad (14)$$

In the isotropic case, the anomalous dimensions γ_N^* in the one-loop approximation are related [14] to anomalous dimensions of composite operators of a simpler model of passively advected scalar field [13], viz. γ_N^* are given by

$$\gamma_N^* = -\frac{N(N+d)\epsilon}{2(d+2)} + O(\epsilon^2), \quad N \geq 2, \quad \gamma_1^* = 0. \qquad (15)$$

From this relation it follows that the scaling exponent in expression (14) $\gamma_N^* - \gamma_{N-m}^* - \gamma_m^* < 0$ at this order. Below it will be shown that these relations are stable against small-scale anisotropy.

In the anisotropic case we will assume that the statistics of the velocity field is anisotropic at all scales and replace the ordinary transverse projection operator in Eq. (8) with the operator

$$T_{ij}(\mathbf{k}) = \left(1 + \alpha_{10}\frac{\mathbf{n}\cdot\mathbf{k}}{k^2}\right)P_{ij}(\mathbf{k}) + \alpha_{20}n_sn_lP_{is}(\mathbf{k})P_{jl}(\mathbf{k}),$$

$$(16)$$

where $P_{ij}(\mathbf{k})$ is the usual transverse projection operator, the unit vector $\mathbf{n}$ determines the distinguished direction, and α_{10}, α_{20} are parameters characterizing the anisotropy. The positive definiteness of the correlation function (8) imposes the following lower bounds on their values $\alpha_{10}, \alpha_{20} > -1$. It should be noted, however, that there are no other restrictions on the parameters α_{10} and α_{20} in relation (16). In particular, they are not supposed to be small but may have arbitrarily large values. For this reason we term this situation as strong anisotropy.

The operator (16) is a special case of the general transverse structure that possesses uniaxial anisotropy

$$T_{ij}(\mathbf{k}) = a(\psi)P_{ij}(\mathbf{k}) + b(\psi)n_sn_lP_{is}(\mathbf{k})P_{jl}(\mathbf{k}), \qquad (17)$$

where ψ denotes the angle between the vectors $\mathbf{n}$ and $\mathbf{k}$ ($\mathbf{n}\cdot\mathbf{k} = k\cos\psi$). Using Gegenbauer polynomials [17] the scalar functions in representation (17) may be expressed in the form

$$a(\psi) = \sum_{i=0}^{\infty} a_i P_{2i}(\cos\psi), \quad b(\psi) = \sum_{i=0}^{\infty} b_i P_{2i}(\cos\psi).$$

For the case of passively advected scalar, it was shown in Ref. [16] that all main features of the general model with the anisotropy structure represented by Eq. (17) are included in the simplified model with the special form of the transverse operator given by Eq. (16). The same argument applies for the present case of passively advected vector field as well.

The uniaxial anisotropy projector (16) has already been widely used in analyzes of the anisotropically driven Navier-Stokes equation, MHD turbulence equations and passive advection equations [18]. However, these studies were limited to the investigation of the existence and stability of the fixed points with the subsequent calculation of the critical dimensions of the basic quantities leaving the calculation of the anomalous exponents in those models an open problem.

The strong small-scale anisotropy (17) affects the inertial-range asymptotic behavior of the correlation functions (14) in two respects: the anomalous dimensions γ_N^* become dependent on the anisotropy parameters α_1 and α_2 and power-like corrections with new anomalous dimensions appear. Combining the results of multiplicative renormalization and OPE in the manner sketched above for the isotropic case, we arrive at the following expression for the inertial-range asymptotics of the correlation functions of the passively advected vector field $B_{N-m,m}$:

$$B_{N-m,m}(r) \sim \nu_0^{-N/2}r_l^N\left(\frac{r_d}{r_l}\right)^{\gamma_N^*+N\epsilon/2}\left(\frac{r}{r_d}\right)^{\gamma_N^*-\gamma_{N-m}^*-\gamma_m^*},$$

$$\underline{\gamma}_1^* = 0, \quad m \geq 1, \qquad (18)$$

with negative exponents $\gamma_N^*+N\epsilon/2 < 0$ and $\gamma_N^* - \gamma_{N-m}^* - \gamma_m^* < 0$. The anomalous dimensions $\gamma_N^*, \underline{\gamma}_{N-m}^*$, and $\underline{\gamma}_m^*$ in Eq. (18) will be defined in relation (48) and results of their numerical

calculation discussed in detail in Sec. IV. The scaling behavior in expression (18) is similar to that of the correlation functions of the passive scalar advected by a *compressible* vector field [14].

III. FIELD-THEORETIC FORMULATION, RENORMALIZATION, AND RG ANALYSIS

The stochastic problem (6) and (8) is equivalent to the field-theoretic model of the set of the three fields $\Phi = \{\mathbf{b}', \mathbf{b}, \mathbf{v}\}$ with the action functional

$$S(\Phi) \equiv \frac{1}{2}\mathbf{b}'D_b\mathbf{b}' + \mathbf{b}'[-\partial_t - (\mathbf{v}\cdot\boldsymbol{\nabla}) + \nu_0\Delta]\mathbf{b} + \mathbf{b}'(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{v}$$

$$- \frac{1}{2}\mathbf{v}D_v^{-1}\mathbf{v}, \qquad (19)$$

where $\mathbf{b}'$ is an auxiliary field (all required integrations over space-time coordinates and summations over the vector indices are implied). The first five terms in Eq. (19) represent the De Dominicis–Janssen action corresponding to the stochastic problem at fixed $\mathbf{v}$ (see, e.g., Refs. [19]), whereas the last term represents the Gaussian averaging over $\mathbf{v}$. D_b and D_v are the correlation functions (7) and (8), respectively.

In this field-theoretic language, the correlation functions (10) are defined as

$$B_{N-m,m}(r) \equiv \int \mathcal{D}\Phi \; b_r^{N-m}(\mathbf{x},t)b_r^m(\mathbf{x}',t)e^{S(\Phi)} \qquad (20)$$

with the action $S(\Phi)$ defined above.

Action (19) is given in a form convenient for application of the quantum-field perturbation analysis with the standard Feynman-diagram technique. The quadratic part of the action determines the matrix of bare propagators. For the fields $\mathbf{b}'$ and $\mathbf{b}$ the propagators in the wave-vector-frequency representation are

$$\langle b_ib'_j\rangle_0 = \langle b'_jb_i\rangle_0^* = \frac{P_{ij}(\mathbf{k})}{-i\omega + \nu_0k^2},$$

$$\langle b_ib_j\rangle_0 = \frac{C_{ij}(\mathbf{k})}{\omega^2 + \nu_0^2k^4},$$

$$\langle b'_ib'_j\rangle_0 = 0, \qquad (21)$$

where $C_{ij}(\mathbf{k})$ is the Fourier transform of the function $C_{ij}(\mathbf{r}/L)$ from Eq. (7). The bare propagator of the velocity field $\langle\mathbf{vv}\rangle_0 \equiv \langle\mathbf{vv}\rangle$ is defined by Eq. (8) with the transverse projector given by Eq. (16). The interaction in the model is given by the nonlinear terms $-b'_i(\mathbf{v}\cdot\boldsymbol{\nabla})b_i + b'_i(\mathbf{b}\cdot\boldsymbol{\nabla})v_i \equiv b'_iV_{ijl}v_jb_l$ with the vertex factor which in the wave-number-frequency representation has the following form:

$$V_{ijl} = i(\delta_{ij}k_l - \delta_{il}k_j).$$

With the use of the standard power counting [5,6] (see also Ref. [16] for peculiarities of rapid-change passive advection models) correlation functions with superficial UV divergences may be identified. These are correlation functions

containing frequency-wave-vector integrals divergent in the limit $\epsilon \to 0$ with divergences brought about by integration over large wave numbers and correspondingly having non-negative wave-number dimension. In the present model superficial divergences exist only in the one-particle-irreducible (1PI) Green function $\Gamma_{\mathbf{b'b}}$. In the isotropic case this Green function gives rise only to the renormalization of the term $\nu_0 \mathbf{b'} \Delta \mathbf{b}$ of action (19) and the corresponding UV divergences may be fully absorbed in the proper redefinition of the existing parameters g_0, ν_0 so that all correlation functions calculated in terms of the renormalized parameters g and ν are UV finite.

When anisotropy is introduced, however, the situation becomes more complicated, because the 1PI Green function $\Gamma_{\mathbf{b'b}}$ produces divergences corresponding to the structure $\mathbf{b'}(\mathbf{n} \cdot \nabla)^2 \mathbf{b}$ in the action of the model [due to peculiarities of the rapid-change models [13] the term $(\mathbf{b'} \cdot \mathbf{n}) \Delta (\mathbf{b} \cdot \mathbf{n})$ possible on dimensional and symmetry grounds does not appear]. The term $\mathbf{b'}(\mathbf{n} \cdot \nabla)^2 \mathbf{b}$ is not present in the original unrenormalized action (19), but has to be added to the renormalized action, therefore the model is not multiplicatively renormalizable. In such a case it is customary to extend the original action (19) by including all terms needed for the renormalization of the correlation functions and thus adding new parameters. As a result the extended model is described by a new action of the form

$$S(\Phi) \equiv \frac{1}{2} \mathbf{b'} D_b \mathbf{b'} + \mathbf{b'}[-\partial_t - (\mathbf{v} \cdot \nabla) + \nu_0 \Delta$$
$$+ \chi_0 \nu_0 (\mathbf{n} \cdot \nabla)^2]\mathbf{b} + \mathbf{b'}(\mathbf{b} \cdot \nabla)\mathbf{v} - \frac{1}{2}\mathbf{v} D_v^{-1} \mathbf{v},$$
$$(22)$$

where a new unrenormalized parameter χ_0 has been introduced.

Of course, the bare propagators (21) of the isotropic model are modified and for the extended action (22) assume the form

$$\langle b_i b'_j \rangle_0 = \langle b'_j b_i \rangle_0^* = \frac{P_{ij}(\mathbf{k})}{-i\omega + \nu_0 k^2 + \chi_0 \nu_0 (\mathbf{n} \cdot \mathbf{k})^2}, \quad (23)$$

$$\langle b_i b_j \rangle_0 = \frac{C_{ij}(\mathbf{k})}{|-i\omega + \nu_0 k^2 + \chi_0 \nu_0 (\mathbf{n} \cdot \mathbf{k})^2|^2}, \quad (24)$$

$$\langle b'_i b'_j \rangle_0 = 0.$$

After this modification all terms needed to remove the divergences are present in action (22), therefore the model becomes multiplicatively renormalizable allowing for the standard RG analysis. The corresponding renormalized action may be written down immediately:
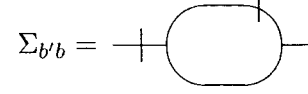

FIG. 1. The (exact) graphical expression for the self-energy operator $\Sigma_{\mathbf{b'b}}$ of the response function of the passive vector field. The plain line denotes the bare propagator (24), and the line with slash (denoting the end corresponding to the arguments of the field $\mathbf{b'}$) corresponds to the bare propagator (23).

$$S_R(\Phi) \equiv \frac{1}{2} \mathbf{b'} D_b \mathbf{b'} + \mathbf{b'}[-\partial_t - (\mathbf{v} \cdot \nabla) + \nu Z_1 \Delta$$
$$+ \chi \nu Z_2 (\mathbf{n} \cdot \nabla)^2]\mathbf{b} + \mathbf{b'}(\mathbf{b} \cdot \nabla)\mathbf{v} - \frac{1}{2}\mathbf{v} D_v^{-1} \mathbf{v}.$$
$$(25)$$

Here, Z_1 and Z_2 are the renormalization constants in which the UV divergent parts of the 1PI response function $\Gamma_{\mathbf{b'b}}$ are absorbed. The renormalized action (25) leads to the multiplicative renormalization of the parameters $\nu_0, g_0,$ and χ_0:

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^\epsilon Z_g, \quad \chi_0 = \chi Z_\chi,$$

where $\nu, g,$ and χ are renormalized counterparts of the bare parameters, and μ is a scale-setting parameter with the same canonical dimension as the wave number. The anisotropy parameters α_{01} and α_{02} are not renormalized, therefore their renormalized counterparts α_1 and α_2 may be put equal to the unrenormalized parameters. In what follows, we will work in the minimal subtraction (MS) scheme, in which—in the one-loop aproximation—renormalization constants have the form $1 + A/\epsilon$, where the amplitude A is a function of $g, \chi, \alpha_1, \alpha_2,$ and d, but independent of ϵ.

Identification of the unrenormalized action (22) with the renormalized one (25) leads to the following relations between the renormalization constants:

$$Z_1 = Z_\nu, \quad Z_2 = Z_\chi Z_\nu, \quad Z_g = Z_\nu^{-1}. \quad (26)$$

It has to be mentioned that the rapid-change models such as Eq. (22) have the nice feature that in all multiloop diagrams of the self-energy operator $\Sigma_{b'b}$ closed circuits of the retarded bare propagators are produced, because the propagator $\langle \mathbf{vv} \rangle_0$ is proportional to the δ function in time. As a result, the one-loop self energy operator $\Sigma_{b'b}$ with the graphical notation of Fig. 1 is exact.

The divergent part of the graph in Fig. 1 is

$$\sum_{b'b}(p) = -\frac{g\nu C_d}{2d(d+2)\epsilon}\{[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2]p^2$$
$$- [2\alpha_1 - (d^2-2)\alpha_2](\mathbf{n} \cdot \mathbf{p})^2\}, \quad (27)$$

where $C_d \equiv S_d/(2\pi)^2$ and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the d-dimensional sphere of unit radius. The expression (27) leads to a straightforward determination of the renormalization constants Z_1 and Z_2:

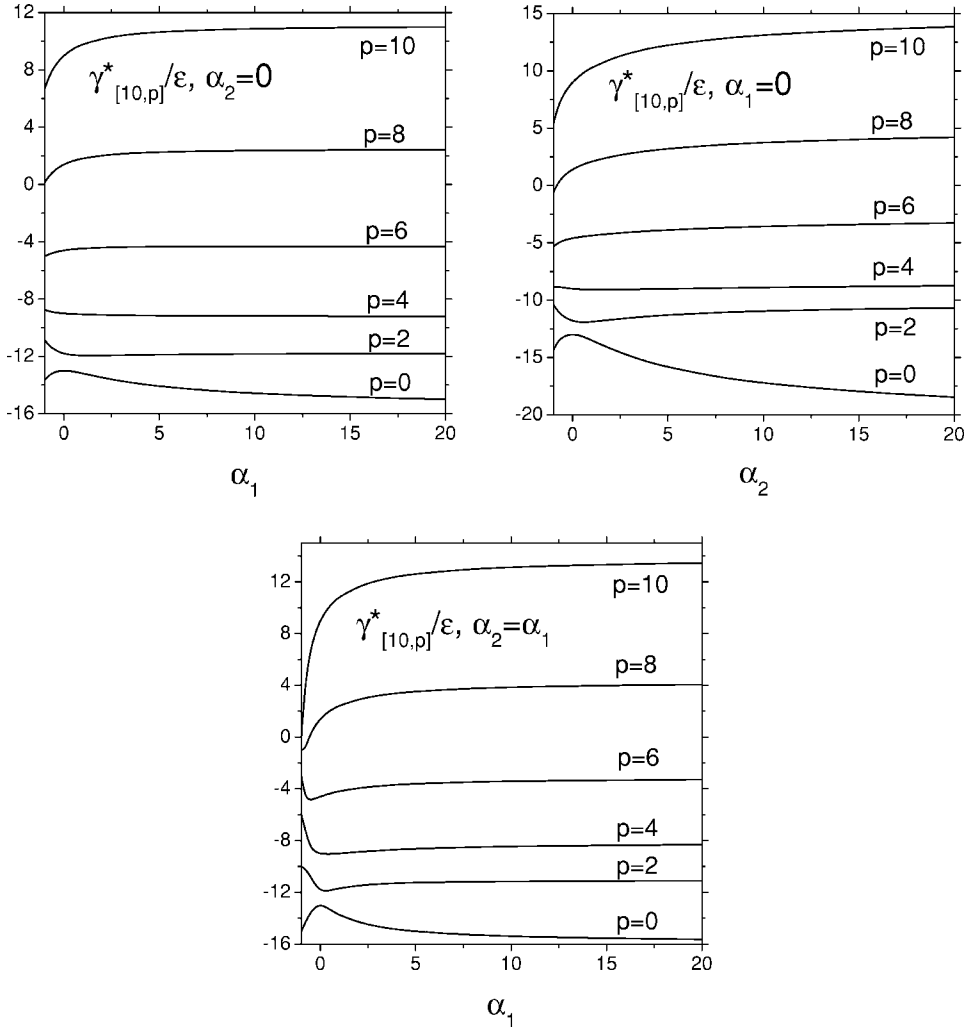$$Z_1 = 1 - \frac{gC_d}{2d(d+2)\epsilon}[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2],$$

FIG. 2. Behavior of the anomalous dimension $\gamma^{*}_{[10,p]}/\epsilon$ in space dimension $d=3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

$$Z_2 = 1 - \frac{gC_d}{2d(d+2)\chi\epsilon}[-2\alpha_1 + (d^2-2)\alpha_2]. \quad (28)$$

With the proper choice of the renormalization constants this renormalization procedure gives rise to UV-finite correlation functions with separate space and time arguments. Independence of the original unrenormalized model of the scale-setting parameter μ of the renormalized model yields the RG differential equations for the renormalized correlation functions of the fields, e.g.,

$$(\mathcal{D}_\mu + \beta_g \partial_g + \beta_\chi \partial_\chi - \gamma_\nu \mathcal{D}_\nu)\langle \mathbf{b}(\mathbf{x},t)\mathbf{b}(\mathbf{x}',t)\rangle_R = 0 \quad (29)$$

with the definition $\mathcal{D}_x \equiv x\partial_x$ for any variable x, and with the following definition of the RG functions (the β functions and the anomalous dimensions γ)

$$\gamma_i \equiv \widetilde{\mathcal{D}}_\mu \ln Z_i$$

for any renormalization constant Z_i and

$$\beta_g \equiv \widetilde{\mathcal{D}}_\mu g = g(-\epsilon + \gamma_1), \quad \beta_\chi \equiv \widetilde{\mathcal{D}}_\mu \chi = \chi(\gamma_1 - \gamma_2), \quad (30)$$

where $\widetilde{\mathcal{D}} \equiv \mu\partial_\mu$ denotes derivative with fixed bare parameters of the extended action (22).

For the anomalous dimensions γ_1 and γ_2 we obtain from Eq. (28)

$$\gamma_1 = \frac{gC_d}{2d(d+2)}[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2], \quad (31)$$

$$\gamma_2 = 1 - \frac{gC_d}{2d(d+2)\chi}[-2\alpha_1 + (d^2-2)\alpha_2]. \quad (32)$$

It should be emphasized that both the renormalization constants (28) and the corresponding anomalous dimensions (31) and (32) in the present model are exact, i.e., they have no corrections of order g^2 or higher.

The fixed points (g_*, χ_*) of the RG equations are defined by the system of two equations

$$\beta_g(g_*, \chi_*) = 0, \quad \beta_\chi(g_*, \chi_*) = 0. \quad (33)$$

It should be noted that as a consequence of relations (26), (30), and (33) at any fixed point with $g_* \neq 0$ the anomalous dimension of viscosity assumes the exact value $\gamma^{*}_\nu = \epsilon$.

The IR stability of a fixed point is determined by the condition that real parts of the eigenvalues of the matrix
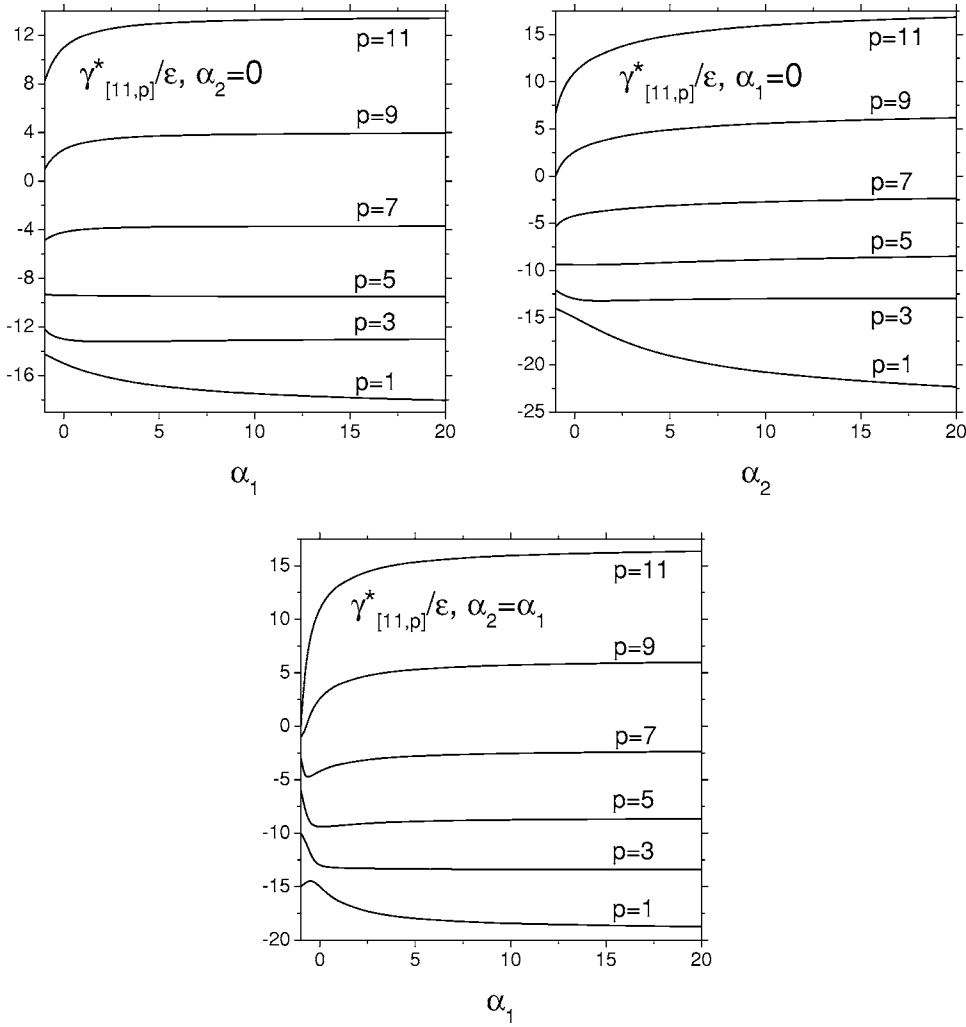
FIG. 3. Behavior of the anomalous dimension $\gamma^*_{[11,p]}/\epsilon$ in space dimension $d=3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

$$\omega = \begin{pmatrix} \partial_g \beta_g & \partial_\chi \beta_g \\ \partial_g \beta_\chi & \partial_\chi \beta_\chi \end{pmatrix}_{\substack{g=g_* \\ \chi=\chi_*}}$$

are positive. Calculation shows that the RG equations have only one nontrivial IR stable fixed point defined by expressions

$$g_* = \frac{2d(d+2)\epsilon}{C_d[(d-1)(d+2) + \alpha_1(d+1) + \alpha_2]}, \qquad (34)$$

$$\chi_* = \frac{-2\alpha_1 + (d^2-2)\alpha_2}{(d-1)(d+2) + \alpha_1(d+1) + \alpha_2}. \qquad (35)$$

Both eigenvalues of the stability matrix ω are equal to ϵ at this fixed point, therefore, the IR fixed point (34) and (35) is stable for $\epsilon > 0$ and all values of the anisotropy parameters α_1 and α_2.

Rather unexpectedly, the β functions and, consequently, the fixed points of the present model of passively advected vector field are exactly the same as in the model of passively advected scalar field [16]. In Sec. IV it will be shown that this similarity is extended to the anomalous scaling dimensions of the composite operators in the OPE representation of the correlation functions as well.

The fixed point (34) and (35) governs the behavior of solutions of Eqs. (29) and the like, and at large scales far from viscous length $r \gg r_d$ at any fixed ratio r/r_l yields the scaling form

$$\langle \mathbf{b}(\mathbf{x},t)\mathbf{b}(\mathbf{x}-\mathbf{r},t) \rangle = D_0^{-1} r^{2-\epsilon} \xi_2(r/r_l), \qquad (36)$$

for the *unrenormalized* correlation function (we remind the reader that due to the absence of field renormalization renormalized and unrenormalized correlation functions are equal but expressed in terms of different variables). It should be noted, however, that the scaling function $\xi_2(r/r_l)$ in Eq. (36) is not determined by the RG Eqs. (29).

This, however, is not enough to find the asymptotic scaling behavior of the two-point correlation functions (10), because they contain products of fields with coinciding spatial arguments. The latter contain UV divergences additional to those included in the renormalization constants (28). These additional divergences due to composite operators (products of fields and their derivatives with coinciding space and time arguments) can be dealt with in a manner similar to that applied to the divergences in the usual correlation functions [6].
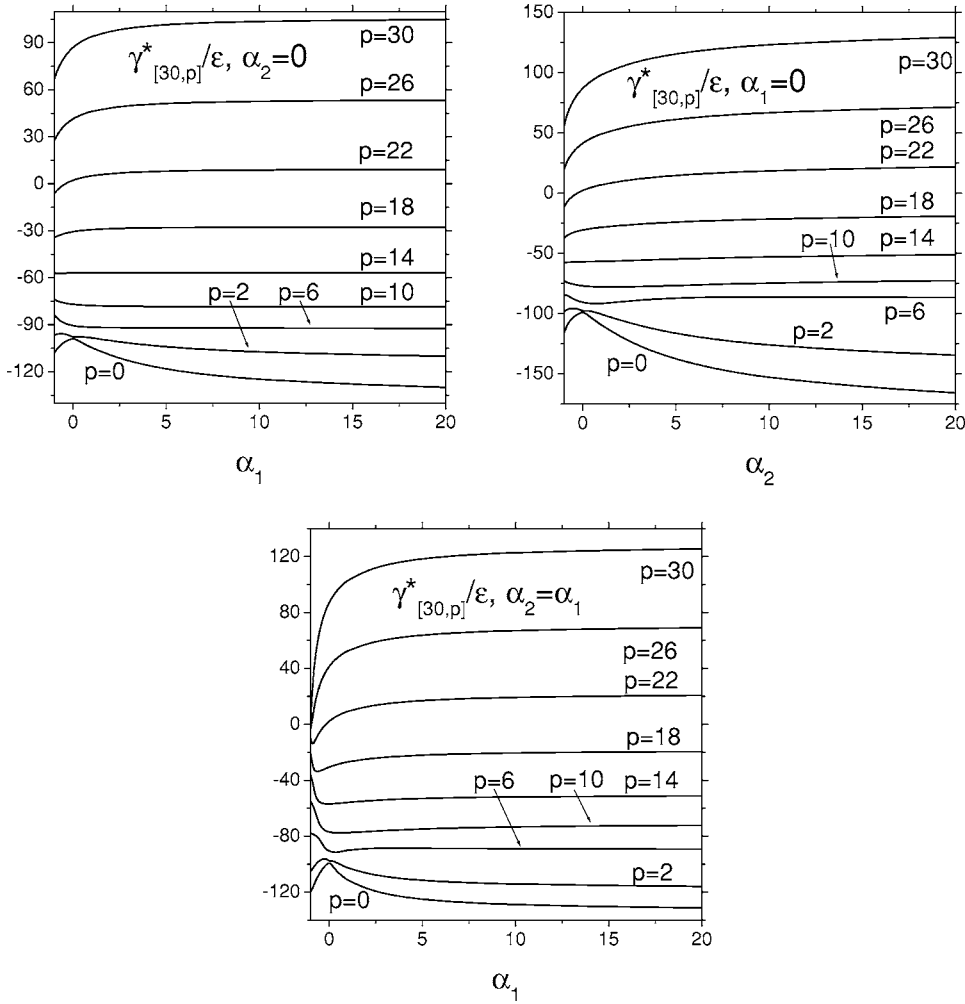
FIG. 4. Behavior of the anomalous dimension $\gamma^*_{[30,p]}/\epsilon$ in space dimension $d=3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

IV. RENORMALIZATION AND CRITICAL DIMENSIONS OF COMPOSITE OPERATORS

A composite operator is any product of fields and their derivatives at a single space-time point $x \equiv (\mathbf{x}, t)$, e.g., $[\mathbf{b}^2(x)]^N$ and $[\partial_i b_j(x) \partial_i b_j(x)]^N$. Two-point correlation functions (20) are averages of products of composite operators at two separate space points. These composite operators are integer powers of the field b_r. Usually, the aim of renormalization of composite operators is to make UV finite all 1PI correlation functions with the insertion of a composite operator F, i.e., quantities of the form

$$\langle F \, \phi_{i_1}(\mathbf{x}_1, t_1) \cdots \phi_{i_m}(\mathbf{x}_m, t_m) \rangle_{1PI}.$$

Correlation functions with such insertions contain additional UV divergences, which also may be removed by a suitable renormalization procedure [5,6]. Composite operators mix in renormalization, therefore an UV finite renormalized operator F^R has the form $F^R = F + \Sigma \Delta F$, where the counterterms $\Sigma \Delta F$ are a linear combination of the operator F itself and, in general, other unrenormalized operators required to make all correlation functions—generated by the renormalized action—with the insertion of F^R UV finite. In this case homogeneous RG equations of the form (29) may be obtained for certain linear combinations of renormalized correlation

functions with composite-operator insertions. Such linear combinations (basis operators $\mathcal{F}$, see below) exhibit IR scaling with definite critical dimensions $\Delta_{\mathcal{F}}$, whereas an arbitrary renormalized composite operator may be expressed as a linear combination of these basis operators.

The general procedure is the following [5–8,13,16]: If $\{F_\alpha\}$ is a closed set of composite operators (i.e., they are mixed only with each other in renormalization), then the sets of renormalized and unrenormalized operators are related through the matrix transformation

$$F_\alpha = \sum_\beta Z_{\alpha\beta} F_\beta^R, \tag{37}$$

$$\gamma_F = Z_F^{-1} \widetilde{\mathcal{D}}_\mu Z_F, \tag{38}$$

where $Z_F \equiv \{Z_{\alpha\beta}\}$ is the renormalization matrix and $\gamma_F \equiv \{\gamma_{\alpha\beta}\}$ is the corresponding matrix of anomalous dimensions for this set of operators. The renormalized composite operators obey the RG differential equations

$$(\mathcal{D}_\mu + \beta_g \partial_g + \beta_\chi \partial_\chi - \gamma_\nu \mathcal{D}_\nu) F_\alpha^R = -\sum_\beta \gamma_{\alpha\beta} F_\beta^R,$$

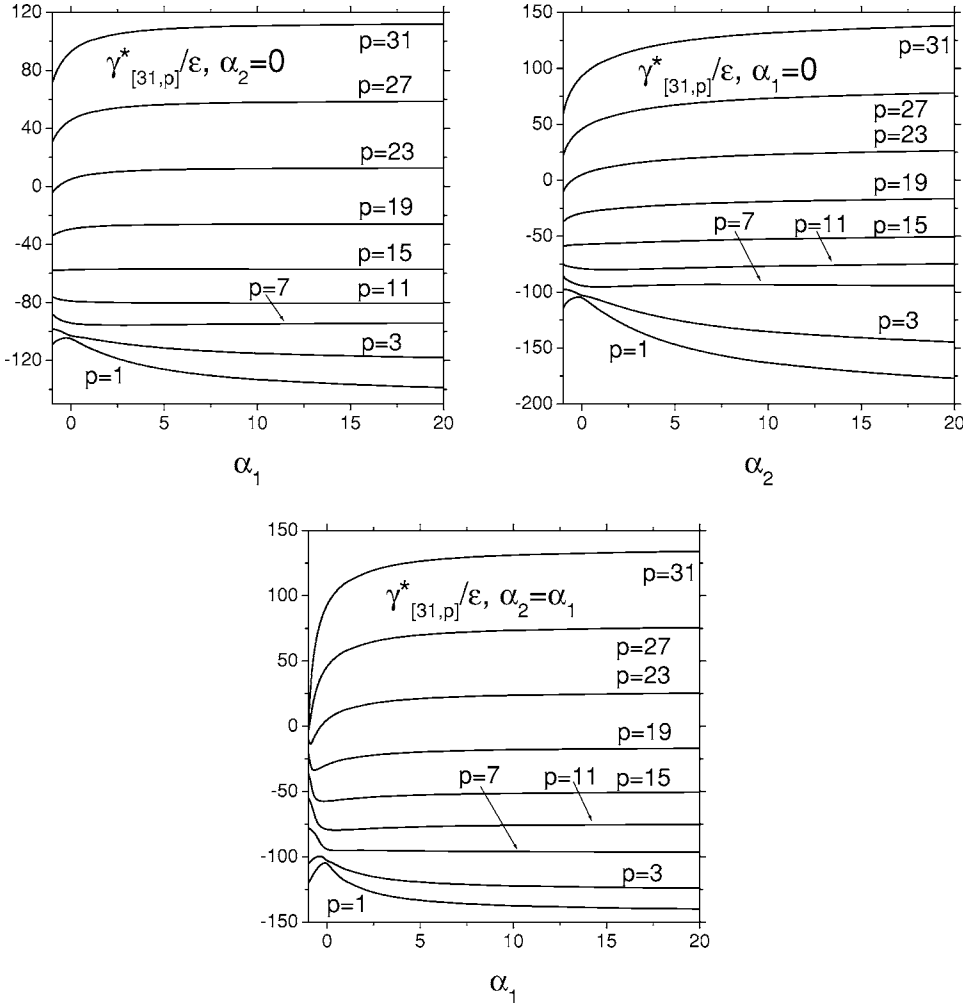which give rise to the matrix of critical dimensions $\Delta_F \equiv \{\Delta_{\alpha\beta}\}$ of the form

FIG. 5. Behavior of the anomalous dimension $\gamma^*_{[31,p]}/\epsilon$ in space dimension $d=3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

$$\Delta_F = d_F^k - \Delta_t d_F^\omega + \gamma_F^*, \quad \Delta_t = -2 + \epsilon, \tag{39}$$

where d_F^k and d_F^ω are, respectively, the diagonal matrices of canonical wave number and frequency dimensions of the operators (where the diagonal elements are sums of the corresponding dimensions of the operators included in the composite operator) and γ_F^* is the matrix of anomalous dimensions (38) at the fixed point (34) and (35).

Critical dimensions of the set $F \equiv \{F_\alpha\}$ are given by the eigenvalues of the matrix Δ_F. The basis operators possessing definite critical dimensions are related to the renormalized composite operators by the matrix transformation

$$\mathcal{F}_\alpha = \sum_\beta U_{\alpha\beta} F_\beta^R, \tag{40}$$

where the matrix $U_F \equiv \{U_{\alpha\beta}\}$ is such that the transformed matrix of critical dimensions $\Delta_{\mathcal{F}} = U_F \Delta_F U_F^{-1}$ is diagonal.

The two-point correlation functions are, however, quantities with insertions of two composite operators. Therefore, it would seem that we would have to consider renormalization of products of two composite operators as well, the aim being then to render UV finite all 1PI correlation functions with two insertions of composite operators. Superficially divergent correlation functions with operator insertions are identified by power counting similar to that of the basic renor-

malization. In the present model such a power counting shows that insertion of products of composite operators of the structure $\mathbf{b}^m(\mathbf{x}, t) \mathbf{b}^n(\mathbf{x}', t)$ does not bring about any new superficial divergences and it is thus sufficient to renormalize the composite operators themselves only in order to make the two-point correlation functions UV finite. Therefore, from the RG analysis of composite operators it follows—by virtue of relations (37) and (40)—that the two-point correlation function $B_{N-m,m}$ may be expressed as a functional average of a quadratic form of basis operators

$$B_{N-m,m}(r) = \sum_{\alpha,\beta} B_{\alpha\beta} \left\langle \mathcal{F}_\alpha\left(\mathbf{x} + \frac{1}{2}\mathbf{r}, t\right) \mathcal{F}_\beta\left(\mathbf{x} - \frac{1}{2}\mathbf{r}, t\right) \right\rangle_R$$
$$\tag{41}$$

with coefficients $B_{\alpha\beta}$ independent of spatial coordinates. Each term in expression (41) obeys the following asymptotic form in the limit $r_d \ll r, r \lesssim r_l$:

$$\left\langle \mathcal{F}_\alpha\left(\mathbf{x} + \frac{1}{2}\mathbf{r}, t\right) \mathcal{F}_\beta\left(\mathbf{x} - \frac{1}{2}\mathbf{r}, t\right) \right\rangle_R$$
$$\sim D_0^{d_\alpha^\omega + d_\beta^\omega} r^{-\Delta_\alpha - \Delta_\beta} r_d^{\gamma_\alpha^* + \gamma_\beta^*} \Xi_{\alpha\beta}\left(\frac{r}{r_l}\right) \tag{42}$$

with the scaling functions $\Xi_{\alpha\beta}$ still to be determined.
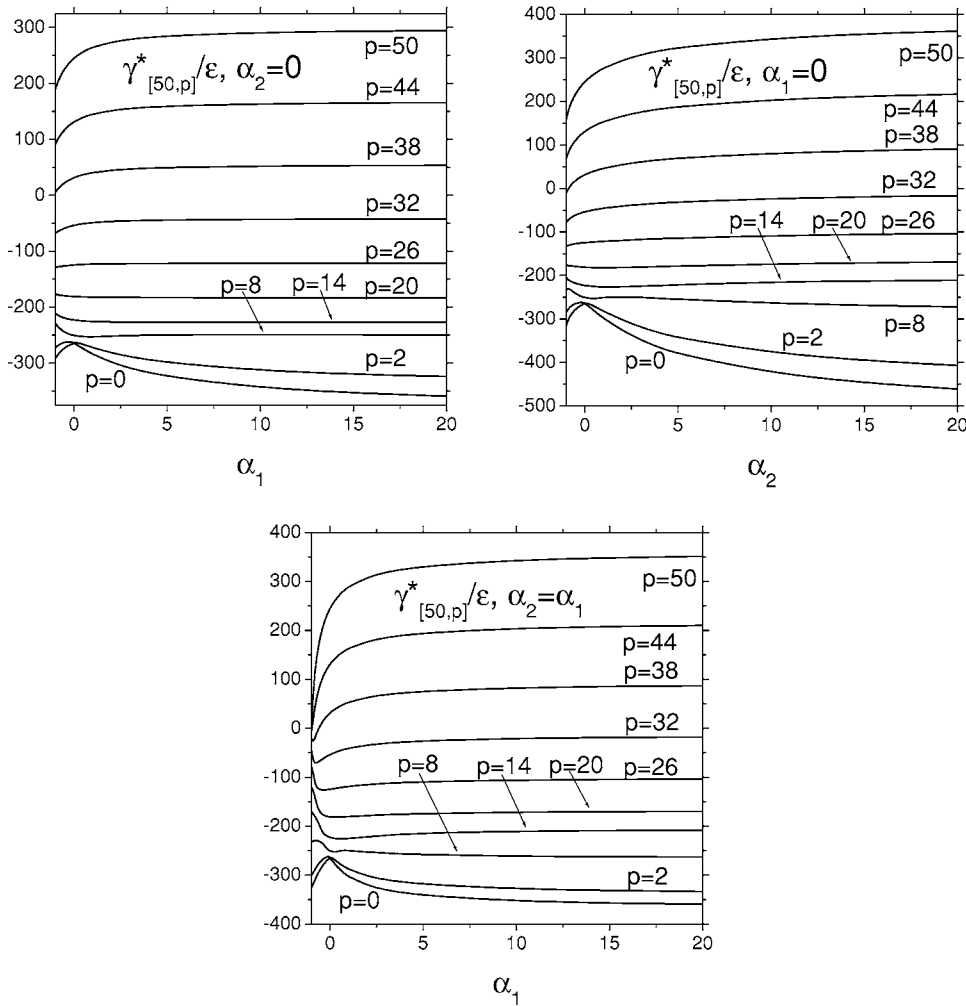
FIG. 6. Behavior of the anomalous dimension $\gamma^*_{[50,p]}/\epsilon$ in space dimension $d=3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

The physically interesting range of scales, however, is the inertial range, specified by the inequalities $r_d \ll r \ll r_l$. The limit $r \ll r_l$ may be explored with the use of the OPE [5,6] as was already discussed in Sec. I. The basic statement of the OPE theory is that the equal-time product of two renormalized composite operators can be represented in the form

$$F^R_\alpha\left(\mathbf{x}+\frac{1}{2}\mathbf{r},t\right)F^R_\beta\left(\mathbf{x}-\frac{1}{2}\mathbf{r},t\right) = \sum_\gamma C_{\alpha\beta\gamma}(\mathbf{r})F^R_\gamma(\mathbf{x},t), \quad (43)$$

where the functions $C_{\alpha\beta\gamma}$ are the Wilson coefficients regular in $1/r_l$, and F^R_γ are renormalized local composite operators which appear in the formal Taylor expansion with respect to $\mathbf{r}$ together with all operators that mix with them in renormalization. If these operators have additional vector indices, they are contracted with the corresponding indices of the coefficients $C_{\alpha\beta\gamma}$.

Without loss of generality we may take the expansion on the right-hand side of Eq. (43) in terms of the basis operators with definite critical dimensions $\Delta_{\mathcal{F}}$. The renormalized correlation function $\langle F^R_\alpha F^R_\beta \rangle_R$ is obtained by averaging Eq. (43) with the weight generated by the renormalized action, the quantities $\langle \mathcal{F} \rangle_R$ appear now only on the right-hand side. Their asymptotic behavior for $r/r_l \to 0$ is found from the corresponding RG equations and is of the form $\langle \mathcal{F} \rangle \propto r_l^{-\Delta_{\mathcal{F}}}$. Com-

parison of the expression for a given function $\langle F^R_\alpha F^R_\beta \rangle_R$ in terms of the IR scaling representation of correlation functions of the basis operators (42) on one hand and the OPE representation brought about by relation (43) on the other in the limit $r_l \to \infty$ allows to find the asymptotic form of the scaling functions $\Xi_{\alpha\beta}(r/r_l)$ in relation (42).

The two-point correlation functions are products of integer powers of the field b_r of the form $b_r^{N-m}(\mathbf{x},t)b_r^m(\mathbf{x}',t)$. Thus, at the leading order in $\mathbf{r}$ their OPE contains operators of the closed set generated by the operator $b_r^N(\mathbf{x},t)$. Power counting and analysis of the structure of graphs shows that this set of composite operators contains only operators consisting of exactly N components of the vector field $\mathbf{b}$, viz. the tensor operators constructed solely of the fields $\mathbf{b}$ without derivatives: $b_{i_1} \cdots b_{i_p}(b_i b_i)^l$ with $p+2l=N$. It is convenient to deal with the scalar operators obtained by contracting the tensor with the appropriate number of the anisotropy vectors $\mathbf{n}$:

$$F[N,p](\mathbf{x},t) \equiv [\mathbf{n} \cdot \mathbf{b}(\mathbf{x},t)]^p [\mathbf{b}^2(\mathbf{x},t)]^l \quad (44)$$

with $N \equiv 2l+p$. Analysis of graphs shows that composite operators (44) with different N do not mix in renormalization, and therefore the corresponding renormalization matrix
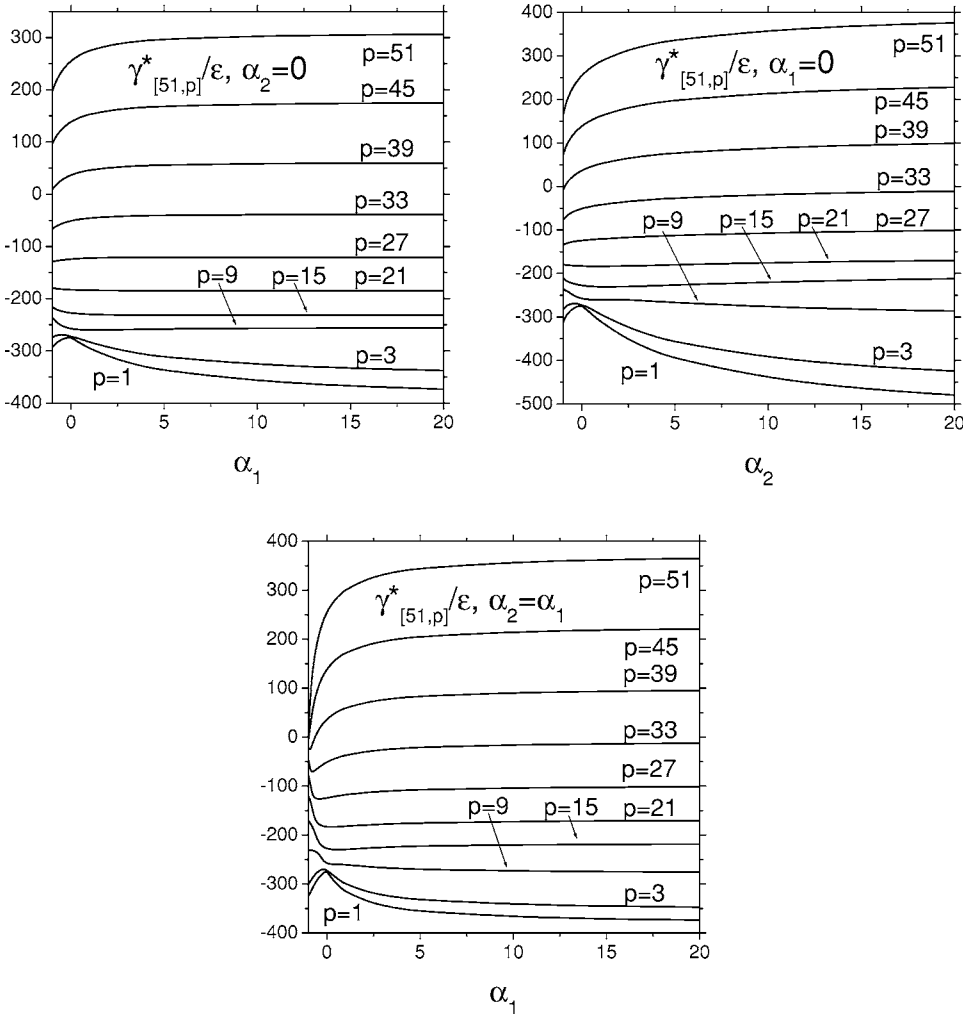
FIG. 7. Behavior of the anomalous dimension $\gamma^*_{[51,p]}/\epsilon$ in space dimension $d=3$ and for representative values of p as functions of anisotropy parameters α_1 and α_2.

$Z_{[N,p][N',p']}$ is in fact block diagonal, i.e., $Z_{[N,p][N',p']}=0$ for $N' \neq N$ and

$$F[N,p] = \sum_{l=0}^{[N/2]} Z_{[N,p][N,N-2l]} F^R[N,N-2l],$$

where $[N/2]$ stands for the integer part of the rational number $N/2$ for odd N (although the odd-order correlation functions $B_{2n+1-m,m}$ vanish, renormalization of the even-order correlation functions $B_{2n-m,m}$ involves odd-order composite operators). Each block with fixed N gives rise to a $(N+1) \times (N+1)$ matrix of critical dimensions whose eigenvalues at the IR stable fixed point are the critical dimensions $\Delta[N,p]$ of the set of operators $F[N,p]$.

Taking into account that renormalization of the composite operators b_r^{N-m} and b_r^m in the correlation function $B_{N-m,m}$ involves operators of the sets $F[N-m,p]$ and $F[m,q]$, respectively, whereas the leading contribution to the OPE involves the set $F[N,s]$, the basis-operator decomposition of the correlation function may be written as

$$B_{N-m,m}(r) \sim v_0^{-N/2} r_d^N \sum_{l_1=0}^{[(N-m)/2]} \sum_{l_2=0}^{[m/2]} \sum_{l_3=0}^{N/2} A_{l_1 l_2 l_3}^{Nm}(r/r_l)$$
$$\times \left(\frac{r}{r_d}\right)^{-\Delta[N-m,N-m-2l_1]-\Delta[m,m-2l_2]} \left(\frac{r}{r_l}\right)^{\Delta[N,N-2l_3]},$$

(45)

where the coefficients $A_{l_1 l_2 l_3}^{Nm}(r/r_l)$ are regular in $(r/r_l)^2$.

The decomposition (45) reveals the inertial-range scaling form of the correlation functions. The leading singular contribution in the limit $r_l \to \infty$, $r_d \to 0$ is given by the basis operator $\mathcal{F}[N,s]$ with the minimal critical dimension $\Delta[N,s]$ and operators $\mathcal{F}[N-m,p]$ and $\mathcal{F}[m,q]$ with the minimal sum of critical dimensions $\Delta[N-m,p]+\Delta[m,q]$. We also remind the reader that the critical dimensions of the basis operators have the structure $\Delta[N,p]=-N(1-\epsilon/2)+\gamma^*_{[N,p]}$, where $\gamma^*_{[N,p]}$ is the anomalous dimension. Therefore, in expression (45) other contributions than the anomalous dimensions cancel in the power of separation distance r. As a result, the correlation functions have asymptotic powerlike behavior as $r/r_l \to 0$ with the minimal anomalous dimension in the basis set generated by the composite operator $b_r^N(\mathbf{x},t)$ and the sum of
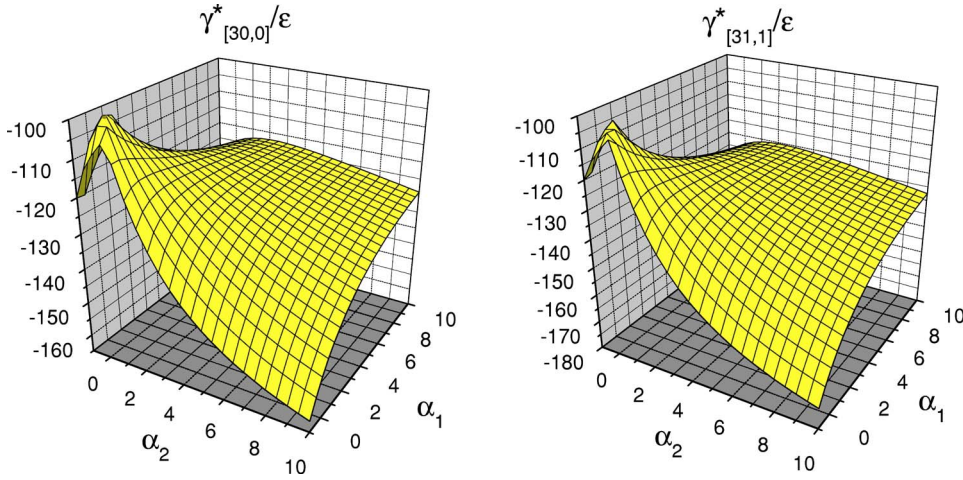
FIG. 8. (Color online.) Behavior of the anomalous dimensions $\gamma^*_{[30,0]}/\epsilon$ and $\gamma^*_{[31,1]}/\epsilon$ in space dimension $d=3$ as functions of anisotropy parameters α_1 and α_2.

minimal anomalous dimensions in the basis set brought about by operators $b_r^{N-m}(\mathbf{x},t)$ and $b_r^m(\mathbf{x},t)$. Calculation shows that these anomalous dimensions grow with the number of field components in the anisotropy direction and thus the minimal anomalous dimension $\underline{\gamma}^*_N$ in any set with fixed N is $\gamma^*_{[N,0]}=\underline{\gamma}^*_N$ for N even and $\underline{\gamma}^*_{[N,1]}$ for N odd. Therefore, the leading asymptotic term of the correlation functions in the inertial range is of the form (we remind the reader that N is an even integer here)

$$B_{N-m,m}(r) \sim v_0^{-N/2} r_l^N \left(\frac{r_d}{r_l}\right)^{\gamma^*_N + N\epsilon/2} \left(\frac{r}{r_d}\right)^{\gamma^*_N - \underline{\gamma}^*_{N-m} - \underline{\gamma}^*_m},$$

$$\underline{\gamma}^*_1 = 0, \quad m \geqslant 1. \tag{46}$$

Numerical results at one-loop order yield negative exponents $\gamma^*_N + N\epsilon/2 < 0$ and $\gamma^*_N - \underline{\gamma}^*_{N-m} - \underline{\gamma}^*_m < 0$, see Figs. 2–7.

A detailed account of practical calculation of the matrix of the renormalization constants $Z_{[N,p][N,p']}$ (which may be readily extended to investigation of all related problems) has been given in Ref. [16] for the advection of a passive scalar, therefore we will not describe all details of the determination of renormalization constants in the present vector model, rather we will discuss its specific features.

It turned out that not only the β functions in the vector and scalar models coincide, but the one-loop renormalization matrices as well. This nontrivial fact stems from the similarities of the mathematical structure of both models. In the model of scalar advection [16] the composite operators $\partial_{i_1}\theta\cdots\partial_{i_p}\theta(\partial_i\theta\partial_i\theta)^l$ constructed solely of the scalar gradients of the scalar admixture θ are needed for calculation of the asymptotic behavior of the correlation functions, whereas in our vector case the main contribution is given by composite operators constructed solely of the fields $\mathbf{b}$ without derivatives. As direct inspection of the relevant diagrams shows, the tensor structures arising upon functional averaging in both cases are in fact identical, which yields the same renormalization matrix $Z_{[N,p][N,p']}$ in both models. Thus, it is not necessary to carry out complete calculations here.

However, in Ref. [16] in the expressions for the general elements of the renormalization matrix of the composite operators there are misprints [for instance, in the definition of the quantity Q_1 in Eq. (76) of Ref. [16] H_4-H_6 should be replaced by H_3-H_6], although the numerical investigation of the critical dimensions is correct. Therefore, we present here the full formulas, in a slightly different form, however.

The only nonzero elements of the matrix $Z_{[N,p][N,p']}$ are

$$Z_{[N,p][N,p-2]} = \frac{gC_d}{d^2-1}\frac{1}{4\epsilon}Q_1, \quad Z_{[N,p][N,p]} = 1 + \frac{gC_d}{d^2-1}\frac{1}{4\epsilon}Q_2,$$
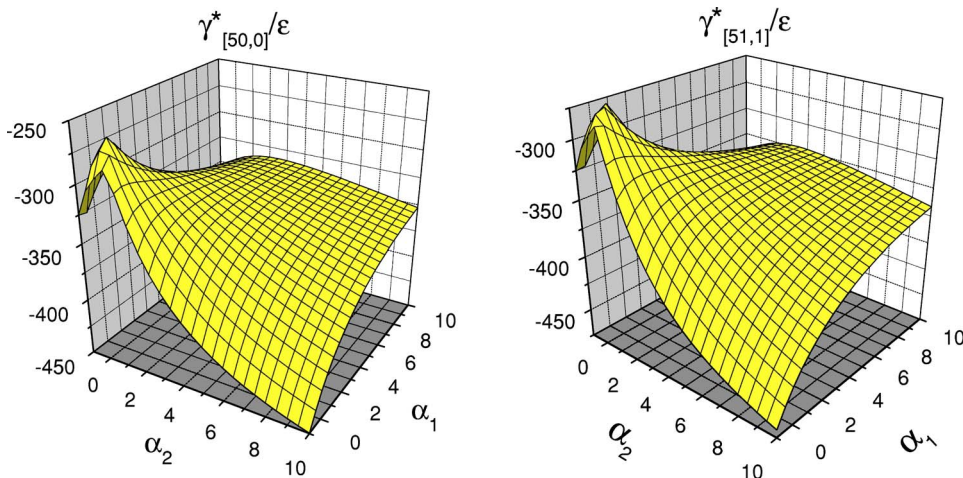

FIG. 9. (Color online.) Behavior of the anomalous dimensions $\gamma^*_{[50,0]}/\epsilon$ and $\gamma^*_{[51,1]}/\epsilon$ in space dimension $d=3$ as functions of anisotropy parameters α_1 and α_2.

$$Z_{[N,p][N,p+2]} = \frac{gC_d}{d^2-1}\frac{1}{4\epsilon}Q_3, \quad Z_{[N,p][N,p+4]} = \frac{gC_d}{d^2-1}\frac{1}{4\epsilon}Q_4,$$

with the coefficients Q_i defined as follows:

$$Q_1 = p(p-1)(d+1)(H_0(1+\alpha_2)+(H_4-H_2)(1-2\alpha_1+3\alpha_2)$$
$$+ H_6(\alpha_1-\alpha_2))$$

$$Q_2 = H_0\{(d-2)(N-p)^2 - (1+\alpha_2)(d+1)(p-1)p + (N-p)$$
$$\times[3+d^2+2d(p-1)+2p+\alpha_2(d+1)(1+2p)]\}$$
$$+ H_2([5+3\alpha_2+\alpha_1(d-2)-d](N-p)^2+(d+1)[1-\alpha_1$$
$$+ d+\alpha_2(d+2)](p-1)p-(N-p)\{9-2d+d^2+8p(d$$
$$+1)-\alpha_1[3+d^2+2d(p-1)+2p]+2\alpha_2[4+d+5p(d$$
$$+1)]\})+H_4(-[3+6\alpha_2+\alpha_1(d-5)](N-p)^2+(d+1)$$
$$\times[\alpha_1(d+1)-\alpha_2(2d+1)-d](p-1)p+(N-p)\{6[1$$
$$+p(d+1)]+\alpha_2[13+d+14p(d+1)]-\alpha_1[9+d^2+8p$$
$$+2d(4p-1)]\})+H_6(\alpha_2+\alpha_1)\{3(N-p)^2+d(d+1)(p$$
$$-1)p-6(N-p)[1+p(d+1)]\},$$

$$Q_3 = H_0(n-p)\{-9-d^2+5N-7p-d(-2+N+p)+\alpha_2(d$$
$$+1)[n-3(1+p)]\}+H_2(N-p)\{d^3+d^2(-2+N+p)$$
$$+6(6-3N+4p)+d(9-5N+13p)+\alpha_2[27-13N$$
$$+21p+d^2(1+2p)+d(16-7N+17p)]+\alpha_1[9+d^2$$
$$-5N+7p+d(-2+N+p)]\}+H_4(N-p)\{-2(2+d)[6$$
$$-3N+(4+d)p]+\alpha_1[d^3+d^2(-2+N+p)+6(6-3N$$
$$+4p)+d(9-5N+13p)]+\alpha_2[48-24N+34p+d^2(1$$
$$+4p)+d(25-12N+26p)]\}+H_62(N-p)(\alpha_2-\alpha_1)(2$$
$$+d)[6-3N+(4+d)p],$$

$$Q_4 = H_0[3+\alpha_2(d+1)](2-N+p)(N-p)+H_2[3\alpha_1-6(2$$
$$+d)-\alpha_2(9+7d+d^2)](2-N+p)(N-p)+H_4(2+d)$$
$$\times[-6\alpha_1+(1+2\alpha_2)(4+d)](2-N+p)(N-p)+H_6(\alpha_1$$
$$-\alpha_2)(8+6d+d^2)(2-N+p)(N-p),$$

where H_i are the functions

$$H_0 = {}_2F_1(1,1/2,d/2,-\chi),$$

$$H_2 = {}_2F_1(1,3/2,d/2+1,-\chi)/d,$$

$$H_4 = {}_2F_1(1,5/2,d/2+2,-\chi)\frac{3}{d(d+2)},$$

$$H_6 = {}_2F_1(1,7/2,d/2+3,-\chi)\frac{15}{d(d+2)(d+4)}.$$

Here, ${}_2F_1$ is the Gauss hypergeometric function. The nontrivial elements of the matrix of anomalous dimensions $\gamma_{[N,p][N,p']}$ are

$$\gamma_{[N,p][N,p-2]} = -\frac{gC_d}{4(d^2-1)}Q_1, \quad \gamma_{[N,p][N,p]} = -\frac{gC_d}{4(d^2-1)}Q_2,$$

$$\gamma_{[N,p][N,p+2]} = -\frac{gC_d}{4(d^2-1)}Q_3, \quad \gamma_{[N,p][N,p+4]} = -\frac{gC_d}{4(d^2-1)}Q_4,$$

(47)

and the matrix of critical dimensions (39) is thus

$$\Delta_{[N,p][N,p']} = -N\left(1-\frac{\epsilon}{2}\right)\delta_{pp'} + \gamma_{[N,p][N,p']}^*,$$ (48)

where the asterisk stands for the value at the fixed point (34) and (35). This represents the critical dimensions of the composite operators (44) at the first order in ϵ. It should to be stressed that in contrast to the value of the fixed point (34) and (35), which has no higher order corrections, the expressions for anomalous dimensions (47) have nonvanishing corrections of order g^2 and higher.

The critical dimensions $\Delta[N,p]=-N(1-\epsilon/2)+\gamma_{[N,p]}^*$ are given by the eigenvalues of the matrix (48). As was already discussed in Ref. [16] in the limiting isotropic case ($\alpha_1=\alpha_2=0$) this matrix becomes triangular, i.e., the eigenvalues are simply equal to the diagonal elements $\Delta[N,p] \equiv \Delta_{[N,p][N,p]}$.

Since our result for the anomalous dimensions is the same as in Ref. [16] for the admixture of a passive scalar, all conclusions about the hierarchical behavior of the critical dimensions of the composite operators are also valid in the analysis of the present model. Nevertheless, the inertial-range asymptotic behavior of the correlation functions in these two problems is completely different, because, first, in the scalar problem single-point products of the scalar are not renormalized, while in the vector problem they are, and, second, the leading contribution to the OPE is given by the products of derivatives of the scalar, whereas in the vector problem products of the field components themselves yield the leading contribution.

In Ref. [16] the behavior of the critical dimensions $\Delta[N,p]$ for $N=2$, 3, 4, 5, and 6 was numerically studied. The main conclusion is that the dimensions Δ_N remain negative in anisotropic case and decrease monotonically as N increases for odd and even values of N separately.

In present paper we concentrate our attention on the investigation of the composite operators (44) for relatively large values of N, namely, we will analyze cases with $N=10$, 11, 20, 21, 30, 31, 40, 41, 50, and 51. Our aim has been to find out whether hierarchies which hold for small values of N remain valid for significantly larger values of N, and the answer turned out to be in the affirmative.

In Ref. [16] several hypothetically possible structures of the matrix of critical dimensions (48) were discussed. In particular, the possibility that the matrix (48) for some α_1 and α_2 would have a pair of complex conjugate eigenvalues $\Delta = \text{Re }\Delta \pm i\,\text{Im }\Delta$ cannot be excluded *a priori*. In this case, the small-scale behavior of the scaling functions would have oscillating terms of the form

$$\left(\frac{r}{r_l}\right)^{\mathrm{Re}\,\Delta} \{C_1\cos[(\mathrm{Im}\,\Delta)r/r_l] + C_2\sin[(\mathrm{Im}\,\Delta)r/r_l]\},$$

with some real constants C_1, C_2.

Another, in general, conceivable structure of the matrix (48) is related to the situation when it cannot be diagonalized but only reduced to the Jordan normal form. In this case, the corresponding contribution to the scaling function would involve a logarithmic correction to the powerlike behavior, viz.

$$\left(\frac{r}{r_l}\right)^{\Delta}[C_1\ln(r/r_l) + C_2],$$

where Δ is the eigenvalue related to the Jordan cell.

In Figs. 2–7 behavior of the eigenvalues of the matrix of anomalous dimensions $\gamma^*_{[N,p]}$ for relatively large values of the N are shown. It can be seen that only real eigenvalues exist in all cases, and also their hierarchical behavior discussed in Ref. [16] is conserved. At first sight the curves for $p=0$ and $p=2$ in the even case and the curves for $p=1$ and $p=3$ in the odd case in Figs. 4–7 appear to be crossing at the point $\alpha_1=\alpha_2=0$ but in fact the curves are only visually running very near together at that point which is a mathematical consequence of the formulas for critical dimensions in the infinitesimal limit $\alpha_1 \to 0$ and $\alpha_2 \to 0$.

In Figs. 8 and 9 the eigenvalues are presented as the functions of two variables α_1 and α_2 for $N=30, 31, 50, 51$ for the first the most singular modes $p=0$ for even N and $p=1$ for odd ones. It can clearly be seen that as values of the parameters α_1 and α_2 increase the anomalous dimensions become more negative approaching some saturated values, therefore the anisotropy amplifies the anomalous scaling.

V. CONCLUSIONS

In this paper we have analyzed asymptotic behavior of the two-point correlation functions $B_{N-m,m}$ of passively advected vector field with small-scale anisotropy. To this end field-theoretic renormalization group and the operator-product expansion have been used in a minimal-subtraction scheme of analytic renormalization.

It is shown that in the inertial range the leading-order powerlike asymptotic behavior of the correlation functions is determined by the critical dimensions brought about already in the isotropic case, which, however, acquire rather strong dependence on the parameters of anisotropy. Powerlike corrections also appear with critical dimensions generated entirely by anisotropic velocity fluctuations. We have calculated numerically the anomalous correction exponents up to order $N=51$ to explore possible oscillatory modulation or logarithmic corrections to the leading powerlike asymptotics, but have found no sign of this kind of behavior: all calculated corrections have had purely powerlike behavior. Our results show that the exponents of the leading powerlike cor-

rections tend to decrease with increasing relative impact of the anisotropy and at large values of the anisotropy parameters approach asymptotic values independent of these parameters. Higher-order corrections for large enough values of the anisotropy index p, however, approach saturation limits through monotonic growth instead of decrease.

From the renormalization-group point of view the present model of passively advected vector field in the presence of strong anisotropy is technically similar to that of passively advected scalar field [16]. This stems from the property that in the scalar case anomalous scaling is brought about by composite operators constructed of gradients of the scalar, i.e., potential vector fields, and the leading-order results of the composite-operator renormalization appear to be insensitive to the nature of the passive vector field. In particular, the β functions and the one-loop contributions to renormalization matrices of relevant composite operators are the same. We have not found, however, any symmetry reason or the like for this to be the case at higher orders, so that the similarity between the cases of potential and solenoidal passive fields may well be a leading-order artifact. Since in the published analysis of the scalar problem [16] there were some misprints, we have also presented corrected complete results of the calculation of the renormalization matrices.

However, there are differences in the scaling behavior in the two cases: instead of the anomalous powerlike growth of the structure functions and magnitudes of the correlation functions in the passive scalar problem in the inertial range, in the present passive vector case a powerlike falloff is predicted for the correlation functions with the subsequent approach to saturation for the structure functions. Moreover, in the scalar problem velocity fluctuation contributions to the inertial-range scaling exponents are (small in ϵ) corrections to the exponents determined by canonical dimensions of the fields, whereas in the vector problem the inertial-range scaling exponents are solely determined by the fluctuation contributions.

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